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Nonlinear wave interactions in shear flows. Part 1. A variational formulation

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A modified version of Bateman's variational formulation of the incompressible Navier–Stokes equations and boundary conditions (see Dryden, Murnaghan & Bateman 1956) is introduced. This is employed to examine a particular nonlinear problem of hydrodynamic stability which was treated previously, using a 'direct' approach, by Craik (1971). This problem concerns the resonant interaction at second order of a triad of wave modes in a parallel shear flow.

The present method is conceptually attractive; it also has the major advantage over the 'direct' method of a substantial reduction in algebraic complexity, which allows results to be derived far more readily. Also, some further improvements are made upon Craik's previous analysis. Such a variational approach may often be simpler than present conventional methods of tackling nonlinear viscous-flow problems. The present paper shows how other problems of nonlinear stability and wave interactions may be tackled in this way.

1. Introduction

In the analysis of weakly nonlinear systems with nearly periodic solutions, variational principles (e.g. Whitham 1967) can lead to overwhelming algebraic simplifications on employing averaging methods (Krylov & Bogoliubov 1947; Mahony 1972). For instance, the variational analysis by Simmons (1969) of resonant interactions among water waves is decisively simpler than earlier analyses (e.g. Phillips 1960; McGoldrick 1965) based directly on Euler's equations of motion; likewise, the analysis of wave interactions in plasmas (Boyd & Turner 1972, 1973) is greatly simplified by averaged Lagrangian methods.

For dissipative systems, such variational techniques are less well developed. In particular, problems of nonlinear hydrodynamic stability are normally dealt with 'directly' using the Navier–Stokes equations. However, Davey (1972), following Benney & Newell (1967), has modified Whitham's theory to incorporate dissipation in a heuristic manner, in order to examine the evolution of a weakly nonlinear wave in a viscous fluid. Whereas the 'direct' method of multiple scales

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applied to the Navier–Stokes equations (e.g. Stewartson & Stuart 1971; DiPrima, Ekhaus & Segel 1971) may be long and intricate, Davey's method affords a simple derivation of the governing equation. Unfortunately, it yields only the *form* of this equation, involving one or more unknown coefficients; to find these coefficients a more complete analysis is required.

For nonlinear 'non-self-adjoint' operators, the construction of variational principles is not well understood (see Finlayson 1972*b*, p. 312). Nevertheless, a variational formulation of the full incompressible Navier–Stokes equations has long been available (Dryden, Murnaghan & Bateman 1956; see also Finlayson 1972*a*, *b*). In this, the variational integral is not a straightforward functional of the velocity vector $\mathbf{u}$ and pressure p [Millikan (1929) and Finlayson (1972*a*) have shown that a variational integral of the latter kind exists only if $\mathbf{u} \times (\nabla \times \mathbf{u})$ or $(\mathbf{u} . \nabla)\, \mathbf{u}$ is zero]. Instead, it involves certain additional dependent variables, which may be thought of as an 'adjoint' or 'image' system, but which have no direct physical significance.

Schechter (1966) reviews various variational formulations for continuous systems and discusses their relationship to the concept of a local potential introduced by Glansdorff & Prigogine (1964). (A local-potential formalism for the full Navier–Stokes equations has apparently not yet been derived, though it seems likely that this could be done without much trouble; the two-dimensional boundary-layer equations are discussed by Schechter.) In discussing the relative merits of the various formulations as means for developing numerical approximations, Schechter strongly favours the local-potential approach and concludes that "the use of 'image' systems as a basis for approximation has been shown to be exceedingly complex and as a consequence is not of significant value. The associated [adjoint] functions must satisfy complicated coupled equations which are devoid of physical meaning. The use of this method is not recommended." These sentiments are echoed by Finlayson (1972*b*), who states that, as a basis for computation, "there appears no important use for [Bateman's] variational principle".

Despite Schechter's and Finlayson's doubts as to its usefulness, a modified version of Bateman's formulation is employed here to examine a problem in nonlinear hydrodynamic stability in which dissipation plays an essential role. This variational approach turns out to be much simpler, algebraically, than the conventional 'direct' analysis of the same problem (Craik 1971); and we believe that this may often be the case in problems of nonlinear wave interactions concerning viscous fluids. To the authors' knowledge, this is the first time that Bateman's variational formulation has been put to practical use.

In the next section, a modified version of Bateman's variational formulation is outlined. Section 3 provides an introduction to the particular problem tackled and §§4 and 5 contain the analysis of this problem, leading to the second-order equations governing the evolution of the respective nonlinear disturbances. An oversimplification present in Craik's previous (1971) analysis is here properly dealt with. A brief appraisal of the variational method is given in §6.

Of course, the variational method has wider applications in nonlinear stability theory than that discussed here. For instance, the restriction in the present work to purely temporal modulation may be removed to deal with the evolution of localized disturbances such as those investigated in the series of papers by Hocking, Stuart and Stewartson (Hocking & Stewartson 1971, 1972; Hocking, Stewartson & Stuart 1972). The particular example treated here serves to demonstrate the potential usefulness of the variational method as an analytical tool, despite current belief in its lack of utility.

2. The variational formulation

We introduce a Cartesian co-ordinate system x_i $(i = 1, 2, 3)$ and denote by u_i and p the velocity vector and pressure of an incompressible fluid of uniform density ρ and viscosity μ. The fluid occupies a region bounded by a fixed surface S, consisting of a part S' on which is prescribed a known distribution F_i of surface force per unit area and a part $S-S'$ on which the velocity vector is given as u_i^p. (Mixed boundary conditions might also be incorporated if required; for example, the normal velocity component and the tangential stresses might be given on some part of the boundary, or part of S might be a deformable fluid surface (see Usher 1974), but such cases are omitted here.) A known body force X_i per unit mass acts throughout.

We shall use a version of Bateman's variational formulation (see Dryden *et al.* 1956; also Finlayson 1972*a*), which will be modified to incorporate all the appropriate boundary conditions. In Cartesian tensor notation, Bateman's variational integral is

$$\left.\begin{aligned} I &\equiv \int_{t_0}^{t_1}\left\{\int_R L\,d\tau + \int_{S'} M_1\,dS\right\}dt,\\ L &\equiv \tfrac{1}{2}E_{ij}(p_{ij}+\rho u_i u_j) + \rho(u_i U_{i,t} + U_j X_j) + P u_{i,i},\\ M_1 &\equiv -\rho l_i U_j u_i u_j - U_i F_i,\\ p_{ij} &\equiv p\delta_{ij} - \mu(u_{i,j}+u_{j,i}), \quad E_{ij} \equiv U_{i,j} + U_{j,i}, \end{aligned}\right\} \qquad (2.1)$$

where δ_{ij} is the Kronecker delta and l_i denotes a component of the unit outward normal at each point of S'. Also, $[t_0, t_1]$ denotes an arbitrary time interval while $d\tau$ and dS are the elements of volume and surface area respectively. Summation over repeated indices is implied and the comma notation is used to denote partial differentiation with respect to the co-ordinates x_i and time t. The quantities U_i and P denote auxiliary functions which will henceforth be referred to as the 'pseudo-velocity vector' and 'pseudo-pressure' but which have no direct physical significance.

Although this variational integral yields the correct equations in R and boundary conditions on S' (as is shown below), it does not incorporate the boundary conditions on $S-S'$. To include these, it is necessary to add a further surface integral, namely

$$\left.\begin{aligned} &\int_{t_0}^{t_1}\int_{S-S'} M_2\,dS\,dt,\\ M_2 \equiv -p_{ij}l_iU_j - \rho l_i U_j[u_i^p u_j &+ u_j^p(u_i - u_i^p)] - \mu\,|\mathbf{u}-\mathbf{u}^p|_{,k}\,\epsilon_{ijk} l_i U_j, \end{aligned}\right\} \qquad (2.2)$$

where ϵ_{ijk} is the permutation tensor.†

† If $S' = 0$ it is necessary that $\int_S u_i^p l_i\,dS = 0$ in order to preserve continuity.

The derivation of this expression for M_2 is complicated by the fact that when $\mu = 0$ the physical boundary condition on $S-S'$ should relate only to the normal velocity component: accordingly, normal and tangential components require separate treatment. The inclusion of these boundary conditions in the variational integral is apparently new; for example, Finlayson merely poses these boundary conditions as additional constraints.

We require I to be stationary ($\delta I = 0$) with respect to independent variations of u_i, p, U_i and P subject to the restrictions that the variations δU_i vanish throughout R at the instants t_0 and t_1 and that the variations δu_i vanish on $S-S'$ for all t. It is readily verified that variations of P and U_i lead to the Navier–Stokes equations

$$\left.\begin{aligned} u_{j,j} &= 0 \quad (\text{from } \delta P) \\ u_{i,t}+u_j u_{i,j} &= -\rho^{-1}p_{,i}+X_i+(\mu/\rho)\nabla^2 u_i \quad (\text{from } \delta U_i)\end{aligned}\right\} \text{ for all } x_i \in R$$

and to the boundary conditions on S'

$$F_i = l_j p_{ij} \quad (\text{from } \delta U_i) \quad \text{for all } x_i \in S',$$

which assert that the stress distribution should balance the given surface force F_i. The variation δP yields no boundary terms on S'.

The variations in p and u_i yield the governing equations and boundary conditions for the image system $\{U_i, P\}$, namely

$$\left.\begin{aligned} U_{j,j} &= 0 \quad (\text{from } \delta p) \\ U_{i,t}+u_j(U_{i,j}+U_{j,i}) &= \rho^{-1}P_{,i}-(\mu/\rho)\nabla^2 U_i \quad (\text{from } \delta u_i)\end{aligned}\right\} \text{ for all } x_i \in R,$$

$$l_i(P-\rho U_j u_j)-l_j(\mu E_{ij}+\rho U_i u_j) = 0 \quad (\text{from } \delta u_i) \quad \text{for all } x_i \in S',$$

and the variation δp yields no boundary terms on S'.

The boundary conditions on $S-S'$ require a little more care. Incorporating the additional integral of M_2, the variation of P still yields no boundary terms, and variations in U_j give

$$\rho l_i(u_i-u_i^p)(u_j-u_j^p)+\mu l_i \epsilon_{ijk}|\mathbf{u}-\mathbf{u}^p|_{,k} = 0 \quad (\text{from } \delta U_j). \tag{2.3}$$

On taking the scalar product with l_j it is seen that

$$l_i(u_i-u_i^p) = 0 \quad \text{for all} \quad x_i \in S-S'$$

independently of whether μ is zero or non-zero. This is the inviscid boundary condition, which prescribes the normal velocity component at each point of $S-S'$. Accordingly, the second term of (2.3) must itself vanish. In vector notation, this is just $\mu(\mathbf{l}\times\nabla|\mathbf{u}-\mathbf{u}^p|) = 0$, which, if $\mu \neq 0$, asserts that $|\mathbf{u}-\mathbf{u}^p|$ remains constant on $S-S'$. If we now introduce the further requirement that $\mathbf{u} = \mathbf{u}^p$ at a *single point* of $S-S'$, we have the boundary conditions for viscous flow that

$$u_i = u_i^p \quad \text{for all} \quad x_i \in S-S'.$$

Variations of p and u_i give the corresponding boundary conditions for U_i on $S-S'$. Variation of p leads immediately to

$$l_i U_i = 0 \quad (\text{from } \delta p), \tag{2.4}$$

corresponding to the inviscid boundary condition for u_i. Variation of u_i, subject to the requirement that $\delta u_i = 0$ on $S-S'$, leads to

$$\mu l_i U_j\{\delta u_{i,j} + \delta u_{j,i} - \epsilon_{ijk}|\delta \mathbf{u}|_{,k}\} = 0 \quad \text{for all} \quad x_i \in S-S',$$

where we have used the fact that the unperturbed u_i is equal to u_i^p. But since $\delta u_i = 0$ on $S-S'$,

$$(\delta u_i)_{,j} = l_j \partial(\delta u_i)/\partial n, \quad |\delta \mathbf{u}|_{,k} = l_k \partial|\delta \mathbf{u}|/\partial n,$$

where n denotes distance along the outward normal to $S-S'$. This yields

$$\mu\,[l_i l_j U_j + U_i]\,\partial(\delta u_i)/\partial n = 0,$$

the third term vanishing identically. Using (2.4), we have

$$U_i = 0 \quad (\text{from } \partial u_i) \quad \text{for all} \quad x_i \in S-S',$$

whenever $\mu \neq 0$. The derivation of the equations and boundary conditions is now complete.

In concluding this section we note that, if the body force X_i is time independent and conservative, so that $X_i = -\Omega_{,i}$, where Ω is a time-independent scalar function, and if U_i and $-P$ are replaced by the physical variables u_i and p, then the expression L is replaced by

$$D(\tfrac{1}{2}\rho u_i u_i - \rho\Omega)/Dt - \Phi,$$

where $D/Dt \equiv \partial/\partial t + u_j \partial/\partial x_j$, $\Phi \equiv \frac{1}{2}\mu e_{ij} e_{ij}$ and e_{ij} is the rate-of-strain tensor. This expression may be recognized as the material time derivative of the kinetic minus potential energy per unit volume less the rate of energy dissipation per unit volume due to viscosity. Some similarity with the classical Lagrangian density is thereby established; but it must be stressed that this expression does *not* yield the Navier–Stokes equations: use of the auxiliary variables is unavoidable.

3. The physical problem

The primary velocity profile is that of a parallel (or quasi-parallel) shear flow

$$\bar{u}_i = [\bar{u}(x_3), 0, 0] \quad (0 \leqslant x_3 \leqslant l)$$

between plane parallel boundaries situated at $x_3 = 0, l$ and of unbounded extent in x_1 and x_2. All variables are taken to be dimensionless: accordingly, l may be regarded as unity for channel flows or infinity for boundary layers. We suppose that this flow is perturbed by three waves with respective x_1, x_2 and t periodicities of the form $\exp i\theta_j$ $(j = 1, 2, 3)$, where

$$\theta_1 = \tfrac{1}{2}\alpha x_1 + \beta x_2 - \tfrac{1}{2}\alpha c_R t, \quad \theta_2 = \tfrac{1}{2}\alpha x_1 - \beta x_2 - \tfrac{1}{2}\alpha c_R t, \quad \theta_3 = \alpha x_1 - \alpha c_R t,$$

α, β and c_R being real constants. If such a wave triad exists, its components will interact resonantly at second order owing to the fact that $\theta_1 + \theta_2 = \theta_3$. Resonant triads of this symmetric form comprise two oblique waves propagating at equal and opposite angles to the x_1 direction and a two-dimensional wave propagating in the x_1 direction. The intensity of these periodic disturbances depends on both x_3 and t.

Craik (1971) has shown that triads of this form are likely to exist for many shear-flow profiles and he gives particular examples for the Blasius boundary layer and for a piecewise-linear boundary-layer profile. Furthermore, it turns out that such triads are of especial interest in nonlinear stability theory, for their interaction is of a particularly powerful kind favouring rapid development of the oblique waves. A full second-order analysis and discussion of this problem are given by Craik. In §§ 4 and 5 Craik's results are rederived via the variational formulation described above in order to demonstrate this technique. In a future paper, the analysis will be extended to third order in wave amplitudes with interesting results.

4. Linear theory

For the perturbed flow, the physical and auxiliary flow variables may be written to a first approximation as

$$\left.\begin{aligned} u_1 &= \overline{u}^0(x_3) + \sum_{j=1}^{3} u_1^{(j)}(\mathbf{x},t), & U_1 &= \sum_{j=1}^{3} U_1^{(j)}(\mathbf{x},t),\\ u_2 &= \sum_{j=1}^{2} u_2^{(j)}(\mathbf{x},t), & U_2 &= \sum_{j=1}^{2} U_2^{(j)}(\mathbf{x},t),\\ u_3 &= \sum_{j=1}^{3} u_3^{(j)}(\mathbf{x},t), & U_3 &= \sum_{j=1}^{3} U_3^{(j)}(\mathbf{x},t),\\ p &= x_1 p^0 + \sum_{j=1}^{3} p^{(j)}(\mathbf{x},t), & P &= \sum_{j=1}^{3} P^{(j)}(\mathbf{x},t), \end{aligned}\right\} \tag{4.1}$$

where p^0 is the imposed longitudinal pressure gradient necessary to sustain the primary flow $\overline{u}^0$ and the superscripts (j) label the respective wave components with periodicities $\exp i\theta_j$. The perturbations are assumed to be small compared with the primary flow and, to a linear approximation, all other wave components and harmonics may be neglected.

After substituting these expressions into the functional I of (2.1) we (at present) formally set to zero all third-order products of perturbation quantities. (In fact, the variational formulation of § 2 strictly applies only to *bounded* domains R, but extension to regions of infinite volume may be effected without much difficulty. Here, the formulation of § 2 is directly applicable, because the periodicity of the flow variables in x_1 and x_2 allows consideration of a finite region R even though the flow domain is of unbounded extent in these co-ordinate directions.) On performing appropriate variations, the linearized equations of motion are recovered for each wave. In particular, variations with respect to $P^{(j)}$ and $p^{(j)}$ yield the continuity equations $u_{j,j}^{(k)} = U_{j,j}^{(k)} = 0$ $(k = 1,2,3)$. For the two-dimensional wave, we incorporate these results into (4.1) by writing

$$\left.\begin{aligned} &u_1^{(3)} = \mathrm{Re}\{D\psi_3\}, \quad u_3^{(3)} = \mathrm{Re}\{-i\alpha\psi_3\}, \quad \psi_3 = \phi_3(x_3)\,A_3(t)\exp i\theta_3,\\ &U_1^{(3)} = \mathrm{Re}\{D\Psi_3\}, \quad U_3^{(3)} = \mathrm{Re}\{i\alpha\Psi_3\}, \quad \Psi_3 = \chi_3(x_3)\,B_3(t)\exp -i\theta_3,\\ &p^{(3)} = \mathrm{Re}\{p_3(x_3)\,A_3(t)\exp i\theta_3\}, \qquad P^{(3)} = \mathrm{Re}\{P_3(x_3)\,B_3(t)\exp - i\theta_3\}, \end{aligned}\right\} \tag{4.2}$$

where $D \equiv \partial/\partial x_3$. Here ψ_3 is a perturbation stream function and Ψ_3 its analogue for the auxiliary variables. The choice of $-i\theta_3$ rather than $i\theta_3$ as the exponent for the auxiliary variables is made for later convenience. The various functions of x_3 and t introduced above are complex valued.

For the oblique waves, we define velocity components $\hat{u}_1^{(j)}$ and $\hat{u}_2^{(j)}$ $(j = 1, 2)$ perpendicular and parallel to the respective wave crests by

$$\gamma u_1^{(1)} = \tfrac{1}{2}\alpha\hat{u}_1^{(1)} - \beta\hat{u}_2^{(1)}, \quad \gamma u_2^{(1)} = \beta\hat{u}_1^{(1)} + \tfrac{1}{2}\alpha\hat{u}_2^{(1)},$$

where $\gamma \equiv (\frac{1}{4}\alpha^2 + \beta^2)^{\frac{1}{2}}$ and the corresponding results for $u_j^{(2)}$ have β replaced by $-\beta$. Note that γ is the total wavenumber of these waves. Again, we incorporate the continuity equations by writing

$$\left.\begin{aligned} &\hat{u}_1^{(j)} = \mathrm{Re}\{D\psi_j\}, \quad u_3^{(j)} = \mathrm{Re}\{-i\gamma\psi_j\}, \quad \psi_j = \phi_j(x_3)\,A_j(t)\exp i\theta_j \\ &\hat{u}_2^{(j)} = \mathrm{Re}\{v_j(x_3)\,A_j(t)\exp i\theta_j\}, \quad p^{(j)} = \mathrm{Re}\{p_j(x_3)\,A_j(t)\exp i\theta_j\} \end{aligned}\right\} \quad (j = 1, 2) \tag{4.3}$$

for the physical variables. The pseudo-velocity and pseudo-pressure are written in the analogous form

$$\left.\begin{aligned} &\gamma U_1^{(1)} = \tfrac{1}{2}\alpha\hat{U}_1^{(1)} - \beta\hat{U}_2^{(1)}, \quad \gamma U_2^{(1)} = \beta\hat{U}_1^{(1)} + \tfrac{1}{2}\alpha\hat{U}_2^{(1)} \\ &\gamma U_1^{(2)} = \tfrac{1}{2}\alpha\hat{U}_1^{(2)} + \beta\hat{U}_2^{(2)}, \quad \gamma U_2^{(2)} = -\beta\hat{U}_1^{(2)} + \tfrac{1}{2}\alpha\hat{U}_2^{(2)} \\ &\hat{U}_1^{(j)} = \mathrm{Re}\{D\Psi_j\}, \quad \hat{U}_3^{(j)} = \mathrm{Re}\{i\gamma\Psi_j\}, \quad \Psi_j = \chi_j(x_3)\,B_j(t)\exp -i\theta_j \\ &\hat{U}_2^{(j)} = \mathrm{Re}\{V_j(x_3)\,B_j(t)\exp -i\theta_j\}, \quad P^{(j)} = \mathrm{Re}\{P_j(x_3)\,B_j(t)\exp -i\theta_j\} \end{aligned}\right\} \quad (j = 1, 2). \tag{4.4}$$

If, as envisaged here, the boundary conditions for the physical quantities are

$$\bar{u}^0 = u_k^{(j)} = 0 \quad (j, k = 1, 2, 3;\ x_3 = 0, l) \tag{4.5}$$

the whole boundary S may be taken as comprising these two planes and S' set to zero.

On substituting in I, still suppressing third-order terms in the perturbation quantities, we may perform variations with respect to the functions ϕ_j, χ_j $(j = 1, 2, 3)$, v_j and V_j $(j = 1, 2)$. The variations with respect to χ_j and V_j should yield the appropriate equations and boundary conditions of linear stability theory. On writing

$$\left.\begin{aligned} A_j^{-1}dA_j/dt &= -B_j^{-1}dB_j/dt \equiv \tfrac{1}{2}\alpha c_I \\ A_3^{-1}dA_3/dt &= -B_3^{-1}dB_3/dt \equiv \alpha\tilde{c}_I \end{aligned}\right\} \quad (j = 1, 2) \tag{4.6}$$

it is indeed found that ϕ_3 satisfies the Orr–Sommerfeld equation and ϕ_1 and ϕ_2 its counterpart for oblique waves, while v_1 and v_2 satisfy equations governing the momentum parallel to the oblique wave crests [see equation (5.4) below and Craik 1971, equations (3.1) and (3.2)]. On the other hand, variations with respect to the ϕ_j establish χ_j as the functions *adjoint* to ϕ_j, satisfying the adjoint Orr–Sommerfeld equation and its oblique-wave counterpart [cf. Craik 1971, equations (3.6)]. (If the exponents of the auxiliary functions had been taken as $+i\theta_j$ then the functions corresponding to the χ_j would satisfy the complex conjugates of the adjoint equations.) In all cases, the appropriate homogeneous boundary conditions are satisfied on $x_3 = 0$ and $x_3 = l$. Finally, variations with respect to v_j yield equations and boundary conditions for V_j which are trivially satisfied on taking V_j to be *identically zero*, and this we can do without loss of generality.

Here, $c_R + ic_I$ and $c_R + i\tilde{c}_I$ are the complex phase velocities of the respective waves. Also, when the physical variables describe a growing wave ($c_I, \tilde{c}_I > 0$) the 'pseudo' variables decay in intensity with time, and vice versa, by virtue of (4.6). The complex phase velocities are determined as eigenvalues associated with the equations and boundary conditions for ϕ_j or χ_j. These eigenvalues depend on α and β, and it is hereafter assumed that values of α and β may so be chosen that the condition for resonance is met: namely, that all three waves have the same value of c_R although c_I and $\tilde{c}_I$ may differ.

5. Second-order theory

In proceeding to second order in wave amplitudes, the variational formulation is used in a manner reminiscent of that which Simmons (1969) developed for resonant gravity-wave interactions in an ideal fluid. Since the perturbations are periodic in x_1 and x_2 we may replace the integrand L in I by its average $\bar{L}$ with respect to x_1 and x_2, thereby suppressing the rapidly oscillating terms which do not contribute significantly to I when R or $[t_0, t_1]$ is large. (Alternatively, the x_1 and x_2 dimensions of R may be chosen as multiples of π/α and $2\pi/\beta$ respectively.) The integral over R is thereby reduced to an integral in x_3 only.

Substituting the expressions (4.1)–(4.4) in L, now retaining third-order terms but discarding all terms with zero average with respect to x_1 and x_2, we have

$$\bar{\bar{L}} \equiv \int_0^l \bar{L}\,dx_3 = \tfrac{1}{2}\operatorname{Re}\Big\{\sum_{j=1}^{2} A_j B_j (I_j + I_{j+3}) + A_3 B_3 I_3 - A_1 A_2 B_3 I_8 - A_2^* A_3 B_1 (I_6 + I_9) - A_1^* A_3 B_2 (I_7 + I_{10}) - \sum_{j=1}^{2} A_j (dB_j/dt)(I_{j+10} + I_{j+13}) - A_3 (dB_3/dt) I_{13}\Big\}, \tag{5.1}$$

where * denotes a complex conjugate and the integrals I_k ($k = 1, \ldots, 15$) are defined as

$$\left.\begin{aligned} I_j &\equiv \int_0^l \chi_j L_1[\phi_j]\,dx_3, \quad I_3 \equiv \int_0^l \chi_3 L_3[\phi_3]\,dx_3, \quad I_{j+3} \equiv \int_0^l V_j G_j\,dx_3, \\ I_6 &\equiv \int_0^l \chi_1 F_{23}\,dx_3, \quad I_7 \equiv \int_0^l \chi_2 F_{13}\,dx_3, \quad I_8 \equiv \int_0^l \chi_3 F_{12}\,dx_3, \\ I_9 &\equiv \int_0^l V_1 G_{23}\,dx_3, \quad I_{10} \equiv \int_0^l V_2 G_{13}\,dx_3, \\ I_{j+10} &\equiv \int_0^l \chi_j L_4[\phi_j]\,dx_3, \quad I_{13} \equiv \int_0^l \chi_3 L_5[\phi_3]\,dx_3, \quad I_{j+13} \equiv -\int_0^l V_j v_j\,dx_3, \end{aligned}\right\} \tag{5.2}$$

where j takes the values 1 and 2 and

$$L_1[\phi_j] \equiv \tfrac{1}{2} i\alpha[(\bar{u}^0 - c_R)(D^2 - \gamma^2) - D^2\bar{u}^0]\,\phi_j - R^{-1}(D^2 - \gamma^2)^2 \phi_j, \tag{5.3a}$$

$$L_3[\phi_3] \equiv i\alpha[(\bar{u}^0 - c_R)(D^2 - \alpha^2) - D^2\bar{u}^0]\,\phi_3 - R^{-1}(D^2 - \alpha^2)^2 \phi_3, \tag{5.3b}$$

$$L_4[\phi_j] \equiv (D^2-\gamma^2)\,\phi_j, \quad L_5[\phi_3] \equiv (D^2-\alpha^2)\,\phi_3, \tag{5.3c,d}$$

$$G_j \equiv R^{-1}(D^2-\gamma^2)\,v_j - \tfrac{1}{2}i\alpha(\overline{u}^0-c_R)\,v_j + (-1)^j\, i\beta D\overline{u}^0\phi_j, \tag{5.3e}$$

$$\begin{aligned} F_{j3} \equiv \tfrac{1}{4}i\alpha\{&(\alpha^2\gamma^{-2}-2)\,\phi_3(D^2-\gamma^2)\,D\phi_j^* + (\alpha^2\gamma^{-2}-3)\,D\phi_3(D^2-\gamma^2)\,\phi_j^* \\ &-2D\phi_j^*(D^2-\alpha^2)\,\phi_3 - \phi_j^*(D^2-\alpha^2)\,D\phi_3 \\ &+(-1)^j\,2\alpha\beta\gamma^{-2}(\phi_3 D^2 v_j^* + D\phi_3 Dv_j^* + \gamma^2\phi_3 v_j^*)\}, \end{aligned} \tag{5.3f}$$

$$\begin{aligned} F_{12} \equiv \tfrac{1}{4}i\alpha\{&D[\phi_1(D^2-\gamma^2)\,\phi_2 + \phi_2(D^2-\gamma^2)\,\phi_1] \\ &-(\alpha^2\gamma^2-2)\,[D\phi_1(D^2-\gamma^2)\,\phi_2 + D\phi_2(D^2-\gamma^2)\,\phi_1] \\ &-2\alpha\beta\gamma^{-2}[v_2(D^2-\gamma^2)\,\phi_1 - v_1(D^2-\gamma^2)\,\phi_2 + D\phi_1 Dv_2 - D\phi_2 Dv_1] \\ &+4\beta^2\gamma^{-2}D(v_1v_2) + 2\beta\alpha^{-1}D^2(\phi_1 v_2 - \phi_2 v_1]\}, \end{aligned} \tag{5.3g}$$

$$\begin{aligned} G_{j3} \equiv \tfrac{1}{4}i\alpha\{&\tfrac{1}{4}(3\alpha^2-20\beta^2)\,\gamma^{-2}v_j^* D\phi_3 - (-1)^j\,\beta\alpha^{-1}(3\alpha^2-4\beta^2)\,\gamma^{-2}D\phi_j^* D\phi_3 \\ &+\tfrac{1}{2}(\alpha^2-4\beta^2)\,\gamma^{-2}\phi_3 Dv_j^* - (-1)^j\,2\alpha\beta\gamma^{-2}\phi_3 D^2\phi_j^* \\ &+(-1)^j\,2\beta\alpha^{-1}\phi_j^* D^2\phi_3\}. \end{aligned} \tag{5.3h}$$

The Reynolds number R of course equals $\rho VL/\mu$, where V and L are the scales of velocity and length chosen for non-dimensionalization.

Inspection of these expressions confirms that the appropriate linear equations for ϕ_j and v_j are recovered on disregarding third-order terms, integrating from t_0 to t_1 and considering variations with respect to χ_j and V_j. In so doing, use is made of (4.6) and the fact that the variations of χ_j and V_j are required to vanish at $t = t_0$ and $t = t_1$. For later reference, we note that these equations are

$$\left.\begin{aligned} L_1[\phi_j] + \tfrac{1}{2}\alpha c_I L_4[\phi_j] &= 0 \\ G_j - \tfrac{1}{2}\alpha c_I v_j &= 0 \\ L_3[\phi_3] + \alpha\tilde{c}_I L_5[\phi_3] &= 0 \end{aligned}\right\} \quad (j = 1, 2) \tag{5.4}$$

and have appropriate homogeneous boundary conditions at $x_3 = 0$ and $x_3 = l$ corresponding to (4·5). It is clear, by symmetry, that we may take $\phi_1 = \phi_2$ and $v_1 = -v_2$.

We write

$$\left.\begin{aligned} dA_1/dt = \tfrac{1}{2}\alpha c_I A_1 + a_1 A_3 A_2^*, \quad dA_2/dt = \tfrac{1}{2}\alpha c_I A_2 + a_2 A_3 A_1^*, \\ dA_3/dt = \alpha\tilde{c}_I A_3 + a_3 A_1 A_2, \end{aligned}\right\} \tag{5.5}$$

where the a_j $(j = 1, 2, 3)$ represent second-order interaction parameters whose determination is the object of the analysis. Using the *linear* estimates for ϕ_j, χ_j, v_j and V_j, the right-hand side of (5.1) may be integrated over the interval $[t_0, t_1]$ to yield

$$\begin{aligned} \tfrac{1}{2}\,\mathrm{Re}\int_{t_0}^{t_1}\{A_2^*A_3B_1(-I_6+a_1I_{11}) + A_1^*A_3B_2(-I_7+a_2I_{12}) + A_1A_2B_3(-I_8+a_3I_{13})\}\,dt \\ -\tfrac{1}{2}\,\mathrm{Re}\left[\sum_{j=1}^{3} A_jB_jI_{j+10}\right]_{t_0}^{t_1}. \end{aligned} \tag{5.6}$$

Here, integration by parts has been used on terms in dB_j/dt while the linear equations (5.4) and their counterparts for the auxiliary functions eliminate several other terms. Considering independent variations of the complex amplitude functions B_j, noting that δB_j must vanish at $t = t_0$ and t_1, we obtain the results

$$a_j = I_{j+5}/I_{j+10} \quad (j = 1, 2, 3), \tag{5.7}$$

where the integrands of I_{j+5} and I_{j+10} are evaluated by linear theory. Indeed, if we wish, we may normalize the linear solutions ϕ_j so that $I_{j+10} = 1$ $(j = 1, 2, 3)$.

These results are equivalent to equations (3.7a, b) of Craik (1971). However, this straightforward derivation and that of Craik are both oversimplifications of the real situation since they do not take into account second-order modifications of the functions ψ_j, Ψ_j, v_j and V_j. In fact, it turns out that results (5.7) are correct, as is seen from the following improved derivation. We write

$$\left.\begin{aligned}
\psi_j &= (A_j\phi_j + A_3A^*_{3-j}\,{}^1\phi_j)\exp i\theta_j + \epsilon_j\\
\psi_3 &= (A_3\phi_3 + A_1A_2\,{}^1\phi_3)\exp i\theta_3 + \epsilon_3\\
\hat{u}_2^{(j)} &= \mathrm{Re}\,(A_jv_j + A_3A^*_{3-j}\,{}^1v_j)\exp i\theta_j + \epsilon_{3+j}\\
\Psi_j &= (B_j\chi_j + B_3A_{3-j}\,{}^1\chi_i + B^*_{3-j}A^*_3\,{}^1\tilde{\chi}_j)\exp(-i\theta_j) + \delta_j\\
\Psi_3 &= (B_3\chi_3 + B_2A^*_1\,{}^1\chi_3 + B_1A^*_3\,{}^1\tilde{\chi}_3)\exp(-i\theta_3) + \delta_3\\
\hat{U}_2^{(j)} &= \mathrm{Re}\,(B_3A_{3-j}\,{}^1V_j + B_{3-j}A^*_3\,{}^1\tilde{V}_j)\exp(-i\theta_j) + \delta_{3+j}
\end{aligned}\right\} \quad (j = 1, 2), \tag{5.8}$$

where the A_i and B_i $(i = 1, 2, 3)$ remain functions of t only, ϕ_i, v_j and χ_i are the linear eigenfunctions of x_3 and ${}^1\phi_i$, ${}^1v_j$, ${}^1\chi_i$, ${}^1\tilde{\chi}_i$, ${}^1V_j$ and ${}^1\tilde{V}_j$ are functions of x_3 representing second-order modifications. The ϵ_k and δ_k represent other second- and higher-order terms which play no part in the present analysis at this order of approximation (but which must be considered in the third-order analysis to be reported in due course).

On substituting (5.8) instead of (4.1)–(4.4) into L, retaining third-order terms, averaging with respect to x_1 and x_2 and integrating over $[t_0, t_1]$ we find that the following additional terms must be added to the right-hand side of (5.6):

$$\tfrac{1}{2}\,\mathrm{Re}\int_{t_0}^{t_1}\{\alpha\tilde{c}_I(B_1A_3A^*_2J_1 + B_2A_3A^*_1J_2) + \alpha(c_I - \tilde{c}_I)B_3A_1A_2J_3\}\,dt + \text{boundary terms}, \tag{5.9}$$

where $\quad J_j \equiv \int_0^l \chi_jL_4[{}^1\phi_j]\,dx_3, \quad J_3 \equiv \int_0^l \chi_3L_5[{}^1\phi_3]\,dx_3 \quad (j = 1, 2).$

Here, as above, we have employed the linear equations to eliminate several groups of terms. In particular, the second-order auxiliary functions ${}^1\chi_i$, ${}^1\tilde{\chi}_i$, ${}^1V_j$ and ${}^1\tilde{V}_j$ make no contribution. The 'boundary terms' arise from integration by parts and are evaluated at the end points t_0 and t_1. Like those in (5.6), the variations of these boundary terms with respect to B_i are identically zero because of the restriction that the variations δB_i must vanish at t_0 and t_1: consequently, they play no part in the subsequent analysis.

We note that for linearly neutral waves $c_I = \tilde{c}_I = 0$ and these additional terms are identically zero, confirming that results (5.7) are correct in this case. When c_I and $\tilde{c}_I$ are non-zero, variations with respect to B_i of expression (5.6) together with these additional terms imply that the estimates (5.7) for a_i ($i = 1, 2, 3$) must be corrected by adding the respective terms

$$-\alpha\tilde{c}_I J_1/I_{11}, \quad -\alpha\tilde{c}_I J_2/I_{12}, \quad -\alpha(c_I-\tilde{c}_I)\, J_3/I_{13}. \tag{5.10}$$

Since the growth rates $\frac{1}{2}\alpha c_I$ and $\alpha\tilde{c}_I$ will typically be small in cases of interest, it is clear that (5.7) will yield very good approximations, even for non-neutral waves.

The integrals J_1, J_2 and J_3 involve the second-order functions ${}^1\phi_i(x_3)$, which have yet to be determined. The equations for ${}^1\phi_i(x_3)$ are readily derived from the variational integral by considering variations in χ_i and equating to zero the resulting terms in $A_2^*A_3B_1$, $A_1^*A_3B_2$ and $A_1A_2B_3$. These are

$$\left.\begin{aligned} L_1[{}^1\phi_1]+\tfrac{1}{2}\alpha c_I L_4[{}^1\phi_1] &= -\alpha\tilde{c}_I L_4[{}^1\phi_1]+F_{23}-a_1L_4[\phi_1],\\ L_1[{}^1\phi_2]+\tfrac{1}{2}\alpha c_I L_4[{}^1\phi_2] &= -\alpha\tilde{c}_I L_4[{}^1\phi_2]+F_{13}-a_2L_4[\phi_2],\\ L_3[{}^1\phi_3]+\alpha\tilde{c}_I L_5[{}^1\phi_3] &= -\alpha(c_I-\tilde{c}_I)\,L_5[{}^1\phi_3]+F_{12}-a_3L_5[\phi_3], \end{aligned}\right\} \tag{5.11}$$

where the left-hand sides correspond to the linear operators of (5.4) and the right-hand sides contain terms in the unknown functions ${}^1\phi_i$, together with non-homogeneous terms which are known from linear theory. (When $c_I = \tilde{c}_I = 0$, results (5.7) are just the conditions for solvability of these equations, which provide the usual means of deriving such interaction coefficients.) Appropriate homogeneous boundary conditions at $x_3 = 0, l$ are satisfied by the ${}^1\phi_i$.

Now, since the terms of (5.11) which are linear in ${}^1\phi_i$ do not correspond exactly to those of (5·4) when $\tilde{c}_I$ and $c_I-\tilde{c}_I$ are non-zero, equations (5.11) possess unique solutions for *every* choice of the constants a_i. In particular, they have unique solutions for the values given by (5.7). Supposing that these choices are made, we may multiply the respective equations by the solutions χ_i of the adjoint linear problems and integrate from 0 to l to obtain the results

$$\alpha\tilde{c}_I\int_0^l \chi_j L_4[{}^1\phi_j]\,dx_3 = \alpha(c_I-\tilde{c}_I)\int_0^l \chi_3 L_5[{}^1\phi_3]\,dx_3 = 0 \quad (j = 1, 2).$$

That is to say, $J_1 = J_2 = J_3 = 0$ and the expressions (5.10), which apparently represented corrections to the a_i, in fact vanish. Accordingly, as claimed above, results (5.7) are *exactly* valid for non-neutral as well as neutral waves.

6. Discussion

We have employed a variational method to determine the second-order equations governing the temporal evolution of a resonant triad of waves in a viscous shear flow, thereby reproducing the results and improving the derivation of Craik (1971). Both conceptually and practically, the variational method has several advantages over the 'direct' method previously used by Craik.

First, there is a substantial reduction in the algebraic manipulations necessary to obtain the governing equations for the respective functions. For example, on starting with the momentum equations, it is necessary to eliminate by cross-differentiation the pressure terms corresponding to each wave component, and this leads to unduly cumbersome expressions for the nonlinear terms, which then require algebraic simplification. In contrast, variations with respect to the appropriate auxiliary functions yield the required equations immediately in the present method.

A further advantage is that the integral expressions for the interaction coefficients a_i arise naturally from the variational formulation, the adjoint linear functions χ_i being identified with the auxiliary 'pseudo-velocity' components. In the 'direct' method the coefficients a_i are normally chosen to satisfy solvability criteria for the non-homogeneous differential equations governing the respective second-order flow components; but this choice is unambiguous only when $c_i = \tilde{c}_i = 0$. For non-neutral waves, the present derivation is preferable.

Here, the evolution equations for the complex wave amplitudes $A_i(t)$ were obtained by considering variations with respect to the complex amplitudes $B_i(t)$ of the auxiliary functions, while the equations for the modal shapes $\phi_i(x_3)$, etc. arose from variations with respect to the modal shapes $\chi_i(x_3)$, etc. of the auxiliary system. This procedure is somewhat similar to that of Simmons (1969), who derived the amplitude equations for resonantly interacting capillary–gravity waves by considering variations with respect to their real amplitudes and phases; since Simmons' problem was self-adjoint, there was no need for him to introduce auxiliary variables. On the other hand, the techniques of Whitham (1967), Benney & Newell (1967) and Davey (1972) involve variations with respect to the wavenumber and frequency of the waves. This latter approach is inappropriate for resonance problems since the waves must remain coupled in phase. But, for non-resonant wave interactions such as the self-interaction of a single weakly nonlinear wave, the two approaches would appear to be equivalent provided that the primary flow depends only on the cross-flow variable x_3. (For an evolving primary flow, Whitham's approach must be used in order to secure a proper local representation for the disturbance.)

A similar variational method may often be employed to advantage in other nonlinear stability problems. For example, Stewartson & Stuart (1971) derived by a 'direct' method the third-order equation governing the evolution in space and time of a two-dimensional disturbance centred on a single wave mode (that which is least stable according to linear theory). Application of the variational method to this problem is straightforward; for, although the analysis must be pursued to third rather than second order and spatial as well as temporal modulation must be incorporated, fewer physical and auxiliary variables are required than in the three-wave interaction examined here. All the results of Stewartson & Stuart may be reproduced via the variational method on introduction of the appropriate scaled variables. The amplitude equations arise from variation with respect to the amplitude of the auxiliary (adjoint) stream function, and explicit expressions for the constants a_1, a_2, d_1 and k in these equations arise naturally without the need to invoke solvability criteria (unlike the simpler variational analysis of Davey (1972), which yields the form of the equations but not these constants). Likewise, with a little more effort, the amplitude equation of Hocking, Stewartson & Stuart (1972) for a three-dimensional disturbance may be rederived.

Like 'direct' analyses, the variational method becomes increasingly complex the more wave modes there are present and the higher the order of approxiation in powers of the wave amplitude; and, because of the need to introduce auxiliary as well as physical variables in the variational method, its advantages may sometimes be outweighed by increased complication in such cases. For instance, in tackling the extension to third order of the resonance problem considered here (when many more variables must be introduced) little advantage is to be gained from the variational method, and we adopt a 'direct' analysis to be reported subsequently.

We have demonstrated the usefulness of a variational approach to nonlinear stability problems and have gained some insight into the significance of the image system in this context. We hope that the extension in § 2 of Bateman's variational formulation for the Navier–Stokes equations may also prove useful in other branches of nonlinear viscous fluid dynamics, despite the long neglect of Bateman's results.

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